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RESEARCH ARTICLE

Mathematical Investigation of Functions

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Abstract

Generally, when the independent variable of a given exponential function is used as an exponent, the function is considered an exponential. Thus, the following can be examples of exponential functions: $f(x) = ab^x + c$, $f(x) = ae^bx + c$, or $f(x) = e^{a^2+bx+c}$. However, deriving functions of these types given the set of ordered pairs is difficult. This study was conducted to derive formulas for the arbitrary constants a ,b, and c of the exponential function $f(x) = ab^x + c$. It applied the inductive method by using definitions of functions to derive the arbitrary constants from the patterns produced. The findings of the study were: a) For linear, given the table of ordered pairs, equal differences in x produce equal first differences in y; b) for quadratic, given the table of ordered pairs, equal differences in x produce equal second differences in y; and c) for an exponential function, given a table of ordered pairs, equal differences in x produce a common ratio in the first differences in y. The study obtained the following forms: $b = \sqrt[d]{r}$, $a = \frac{q}{b^n(b^d-1)}$, $c = p - ab^n$. Since most models developed used the concept of linear and multiple regressions, it is recommended that pattern analysis be used specifically when data are expressed in terms of ordered pairs.

KEYWORDS:
Patterns, Patterns of Functions, Pattern Analysis, Models, Exponential Models, Derivation of Functions

1 | INTRODUCTION

The exponential function is defined mathematically as f with a base a denoted by $f(x) = a^x$ where a>0, a$\neq$ 1, and x is any real number. (Alferez and Duro, 2001). Given the set of ordered pairs, {(0, 1), (1, 2), (2, 4), (3, 8), (4, 16)}, the equivalent exponential equation $f(x) = 2^x$ can be derived either by inspection or substitution.

Other than $f(x) = a^x$, exponential functions can come in different forms such as $f(x) = ab^x + c$, $f(x) = ae^{bx} + c$, or $f(x) = e^{ax^2+bx+c}$, and deriving functions of these types given the set of ordered pairs is difficult and complicated.

There are existing mathematical models that are used to predict future outcomes. For example, the model $P_n = P_o(1+r)^n$, is used to predict the population before or after n years; the double-life of bacteria in a culture is determined using the form, $P_n = P_o(2)^{n/k}$; the half-life of a substance is computed using $P_n = P_o(\frac{1}{2})^{n/k}$; when a certain amount of money is invested at an interest that is compounded continuously the form $A_n = A_oe^{rt}$ is used; and, the purchasing power of peso depends on the model $P_n = P_o(1+r)^n$.

According to Shukla et. al (2011), the exponential model, $y = \alpha - \beta\rho^x$, has been very popular among researchers of different disciplines using nonlinear models. It has been extensively used in growth study wherever phenomenon exhibits asymptotic behavior. Kalman (2001), who was interested in using generalized logarithm (g-log) in solving exponential-linear equations, mentioned $c_n = ab^n + d$ as the consumption model of world petroleum reserves, where c_n represents the amount of the resource consumed in the nth interval, and a, b, and $d$are numerical constants.

The study conducted by Stevens (1951) and Patterson (1953) focused on the two models, $y = \alpha + \beta\rho^x$ and $y = \alpha - \beta\rho^x$. In general, their derivations of the parameters made use of regression and linear algebra with the application of differential equations. In addition, their studies used a different number of observations; Stevens used 3 while Patterson used 4.

Exponential functions can be in the form $f(x) = ab^x$. Models of this form are not difficult to deal with because there are existing solutions that can be used to solve problems of this type. In fact, Liu and Spiegel (1999), suggested the formulas $a = antilog\ a'$, and $b = antilog\ b'$ since $f(x) = y = ab^x$ can be expressed in logarithmic form $log\ y = log\ a + (log\ b)x$ in deriving the function. Another mathematician, by the name of William Cherry, used the linear relationship of two points from the graph of an exponential function of the form $f(x) = ab^x$. He concluded that a can be evaluated using $a = (\frac{y_2}{y_1})^{\frac{1}{x_2 - x_1}}$. After citing the methods used and developed by some mathematicians, book authors, and educators, it can be noted that mathematics has abundant concepts and tools to be used in solving or deriving functions, and eventually, developing models.

Most models are developed using regression analysis with the aid of computer programs like SPSS, Minitab, E-Views, STATA, PHStat and some others. However, when relationships between two variables in the form of ordered pairs are given, pattern analysis may work in developing models. To serve as guide in showing how possible pattern of a function and formulas for arbitrary constants of exponential function could be derived, the study included derivations of the arbitrary constants of n-degree functions given the sets of ordered pairs. In this study, n-degree functions referred to linear and quadratic functions.

Results of this study will help book writers/authors and fellow math educators in preparing their manuscripts involving patterns of functions and derivation of exponential function of the form $f(x) = ab^x + c$. To the teachers, this study will provide specific information about the derivation of exponential function specifically of the form $f(x) = ab^x + c$. The formulas for the arbitrary constants *a, b,* and *c* will be derived by the researcher comprehensively so that the transfer of knowledge to the prospective beneficiaries will be less difficult.

Future researchers who may desire to develop a different process of deriving formulas for arbitrary constants a, b, and c of the exponential function of the form $f(x) = ab^x + c$ will benefit from the results of this study.

Specifically, this study sought answers to the following questions: a) What are the patterns of linear, quadratic, and exponential function? b) What are formulas in finding the values of the arbitrary constants a, b, and c of the exponential function $f(x) = ab^x + c$.

The study started from citing the standard forms of linear, quadratic, and exponential functions which are $f(x) = ax + b$, $f(x) = ax^2 + bx + c$, and $f(x) = ab^x + c$, respectively. By substituting the values n, $n + d$, $n + 2d$, $n + 3d$, and $n + 4d$ to each of the given functions, and the differences between any two consecutive functional values are taken, distinct patterns were identified. These patterns were used to derive the formulas for the arbitrary constants. After the different formulas were developed, apparent proofs were given in order to establish its usability.

2 | METHODOLOGY

The study applied the inductive method of research. The process started from using the definitions of linear and quadratic functions. This was followed by establishing tables of ordered pairs using the three standard forms $f(x) = ax + b$, $f(x) = ax^2 + bx + c$, and $f(x) = ab^x + c$. The equally-spaced values $n, n + d, n + 2d, n + 3d$, and $n + 4d$ were then substituted in each of these functions to see whether possible recognizable patterns could exist or not. When patterns were evident, the formulas for finding the values of the arbitrary constants could be derived.

With the aid of the definition, the set of first differences between any two consecutive functional values were analyzed. Similarly, the set of second differences between any two consecutive functional values were analyzed.

The main purpose of this study was to develop formulas for arbitrary constants a, b, and c of the function $f(x) = ab^x + c$.

3 | RESULTS AND DISCUSSIONS

There are different types of functions. When they are expressed in terms of ordered pairs, they can easily be identified because of the patterns that can be derived using the values of the independent and dependent variables. For that purpose, the following functions are considered:

3.1 | Linear Function

Definition: A function, where, given a table of ordered pairs, equal differences (d) in the independent variable (x) produce equal first differences in the dependent variable (y).

The standard form of linear function is $f(x) = ax + b$. Given the values of x which are n, $n+d$, $n+2d$, $n+3d$, and $n+4d$, the functional values can be obtained by substitution.

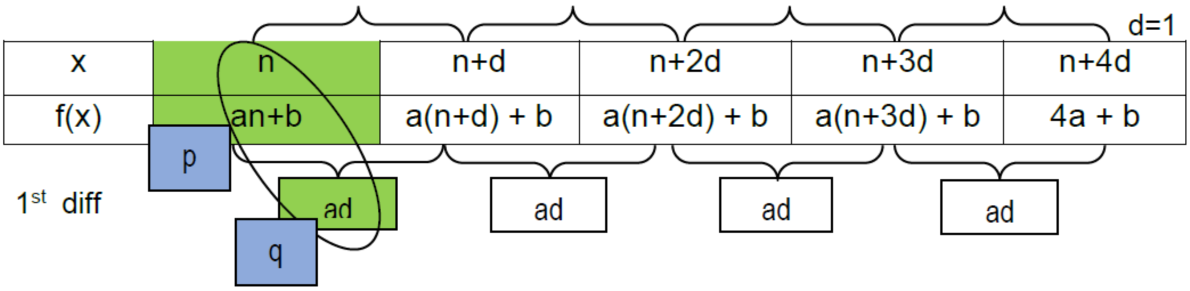

FIGURE 1 Pattern of Linear Function

Evident from the Figure 1 are the values of the independent variable (x) which are equally-spaced. Notice that when the values of x are equally-spaced, the first differences produced in $f(x)$ are equal. In this case, the given function has a linear trend. Now, the problem is how to derive the values of the constants. To do this, a particular value of x is chosen from the table, say n. Then to solve for $a$and$b$, let $p = an + b$, and $q = ad$. For example, derive the function if it is known that the relation, {(0,5), (1,7),(2,9), (3,11), (4, 13)}, has a linear trend. Figure 2 shows the relation and the common difference, d=1.

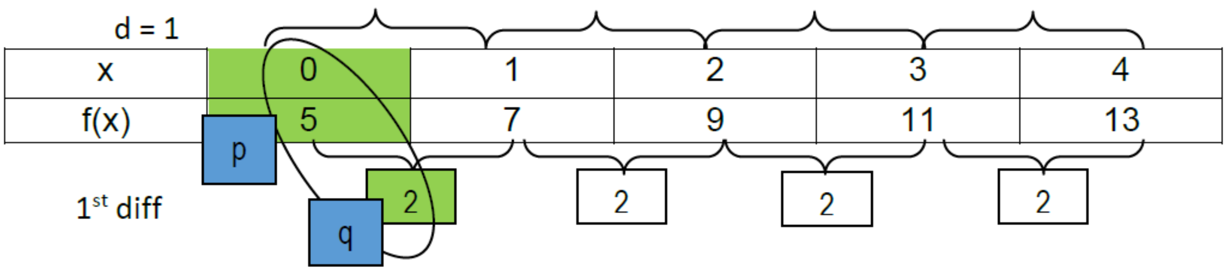

FIGURE 2 Example of a Function of Linear Trend

Evidently from the $f(x)$, the values of the first differences are equal. Hence, the given relation describes a linear function. To derive the function, use $p = an + b$ and $q = a$. Under $x = 0$, $d = 1$, $p = 5$, and $q = 2$, hence, $a = 2$, and $b = 5$. Substituting the values of a and b in $f(x) = ax + b$, the function is $f(x) = 2x + 5$.

Another way to derive the linear function is by solving a system of equations. This is done by forming two equations using the ordered pairs. Using the two ordered pairs, (1, 7) and (2, 9) from the given set, the following equations of the form $f(x) = ax + b$ are derived:

$$\text{At } x = 1, \text{ and } f(x) = y = 7 \rightarrow 7 = a(1) + b \text{ or } a + b = 7 \tag{1}$$

$$\text{At } x = 2, \text{ and } f(x) = y = 9 \rightarrow 9 = a(2) + b \text{ or } 2a + b = 9 \tag{2}$$

By elimination method, a = 2, and b = 5. Hence, the linear function is $f(x) = 2x + 5$.

3.2 | Quadratic Function

Definition: A function, where, given a table of ordered pairs, equal differences (d) in the independent variable (x) produce equal second differences in the dependent variable (y).

To derive the formulas for the arbitrary constants a, b, and c, the standard form $f(x) = ax^2 + bx + c$ is used. Figure 3 shows that the values of the independent variable x are equally spaced, evidently with d=1. Notice the pattern, when the values of x are equally spaced, the second differences produced in the $f(x)$ are equal. In this case, the given function has a quadratic trend.

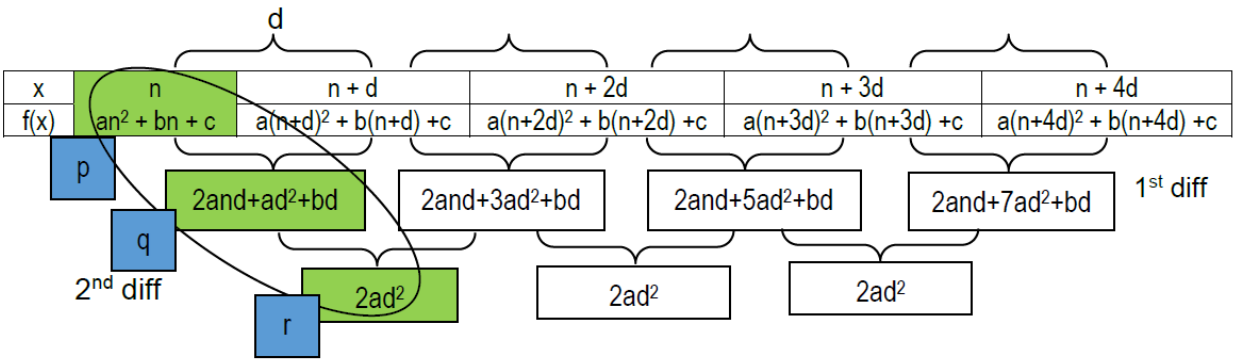

FIGURE 3 Pattern of Quadratic Function

Now, the problem is how to derive the values of the constants. To do this, a particular value of x is chosen from the table, say 0. Then to solve for a, b, and c, let p=an^2 + bn + c, q = 2and + ad^2 + bd, and r = 2ad^2. Inherent to the assumption, $a = \frac{r}{2d^2}$, b=$\frac{q-2and-ad^2}{d}$, and $c = p - an^2 - bn$. These formulas can now be used to model quadratic functions under different conditions.

3.2.1 | Examples

- **CASE 1:** $d = 1$.
  Evidently from the f(x), the values of the second differences are equal as shown in Figure 4 . Hence, the given relation

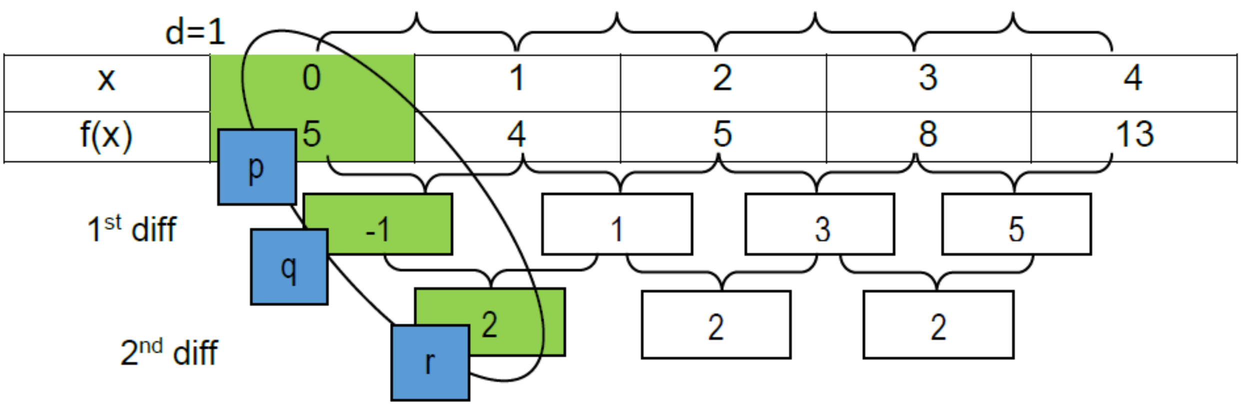

FIGURE 4 Example of a Function of Quadratic Trend, d=1

describes a quadratic function. To derive the function, the established formulas for a, b, and c can be used. Under $x = n = 0$, $p = 5$, $q = -1$, and $r = 2$. By substitution, $a = \frac{2}{2(1)^2} = 1$, $b = \frac{-1-2(1)(0)(1)-1(1)^2}{1} = -2$, and $c = 5 - (1)(0)2 - (-2)(0)5$.

Hence, the function is $f(x) = x^2 - 2x + 5$. The checking can be done by using a particular value of x from the given set, say 2. Under $x = n + 2d = 2$, $p = 5$, $q = 3$, and $r = 2$. By substitution, a $= \frac{2}{2(1)^2} = 1$, $b = \frac{3-2(1)(2)(1)-1(1)^2}{1} = -2$, and $c = 5 - (1)(2)^2 - (-2)(2) = 5$. Hence, the function is $f(x) = x^2 - 2x + 5$.

- **CASE 2:** $d \neq 1$.
 Since the second difference are equal, hence the function has a quadratic trend as shown in Figure 5 . Hence, the function

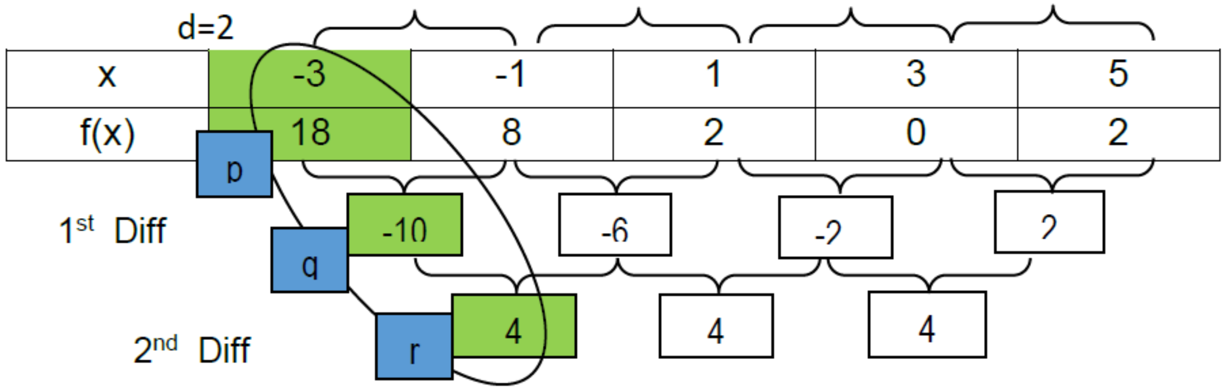

FIGURE 5 Example of a Function of Quadratic Trend, d=2

has a quadratic trend. To derive the function, we use the values for p, q, and r. Under $x = -3$, $p = 18$, $q = -10$, and $r = 4$. By substitution, $a = \frac{2}{(2)^2} = \frac{1}{2}$, $b = \frac{-10-2\left(\frac{1}{2}\right)(-3)(2)-(\frac{1}{2})(2)^2}{2} = -3$, and $c = 18 - (\frac{1}{2})(-3)^2 - (-3)(-3) = \frac{9}{2}$. Hence, the function $f(x) = \frac{1}{2}x^2 - 3x + \frac{9}{2}$.

3.3 | Exponential Function

One of the forms of exponential function is $f(x) = ab^x + c$. Graphing an exponential function of this form is very easy but deriving it given the set of ordered pairs under different conditions is difficult. One must be able to compute first for the correct values of a, b, and c given the set of ordered pairs before a model can be created. But this task requires a number of trials to develop the excellent technique.

For the purpose of facility, the researcher will derive the formulas for the three arbitrary constants of a, b, and c of the exponential function $f(x) = ab^x + c$. First, we set up a table as shown on Figure 6 the relationship of x and y given $f(x) = ab^x + c$.

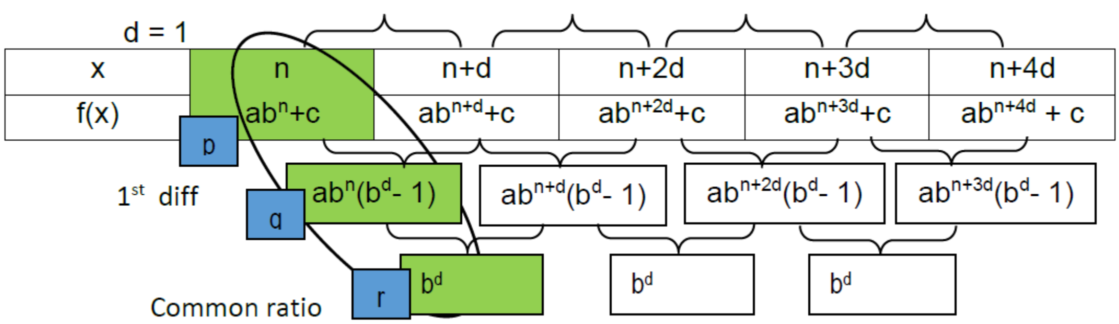

FIGURE 6 Pattern of Exponential Function f(x) = abx + c

Using the equally-spaced values of $x$in terms of n and d are any real numbers, and d denotes the common difference, the values of the functions are $ab^n + c$, $ab^{n+d} + c$, $ab^{n+2d} + c$, $ab^{n+3d} + c$, and$ab^{n+4d} + c$, and so on. Using these values, the first difference between any two consecutive values of f(x) is taken. These values produce the common ratio of b^d.

Now, to derive the formulas for the arbitrary constants, any particular value of independent variable x can be chosen. Assuming a value x = n, and letting p be equal to the value of the exponential function when x = n and n + d; and r be the common ratio produced by the differences, then $p = ab^n + c$, $q = ab^n(b^d - 1)$, and$r = b^d$

Inherent to the above assumptions, the following formulas are obtained:

$$b = \sqrt[d]{r},\ a = \frac{q}{b^n(b^d - 1)},\ c = p - ab^n \tag{3}$$

Apparent Proofs: Given the table of ordered pairs, if it is known that the function is of the form $f(x) = ab^x + c$, derive the function.

- **CASE 1:** $d = 1$.

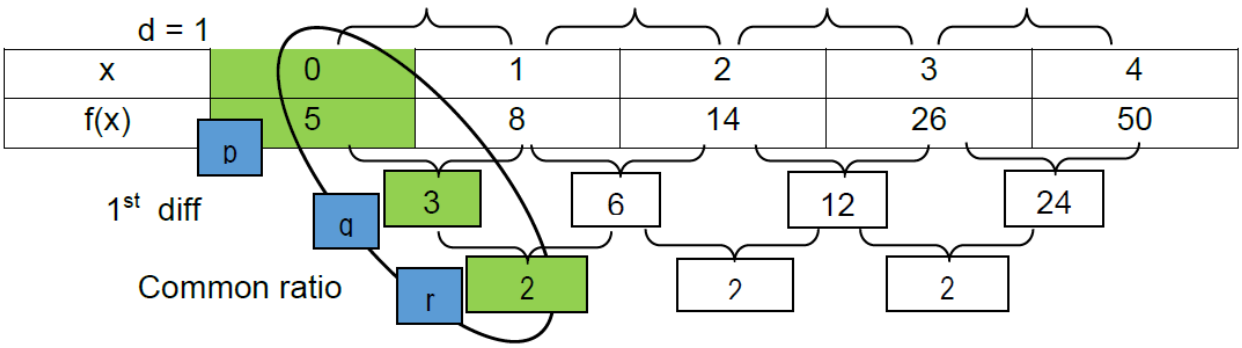

FIGURE 7 Example of a Function of Exponential Trend, d=1

Solution: Evidently, from Figure 7 , $d = 1$. At $x = 2$, $p = 8$, $q = 6$, and $r = 2$. Using the formulas, $b = \sqrt[d]{r} = \sqrt[1]{2} = 2^{1/1} = \mathbf{2}$, $a = \frac{q}{b^x(b^d-1)} = \frac{12}{2^2(2^1-1)} = \mathbf{3}$, and $c = p - ab^x = 14 - 3(2)^2 = \mathbf{2}$. Hence, the function described by the table of ordered pairs is $f(x) = 3(2)^x + 2$.

- **CASE 2:** $d \neq 1$.

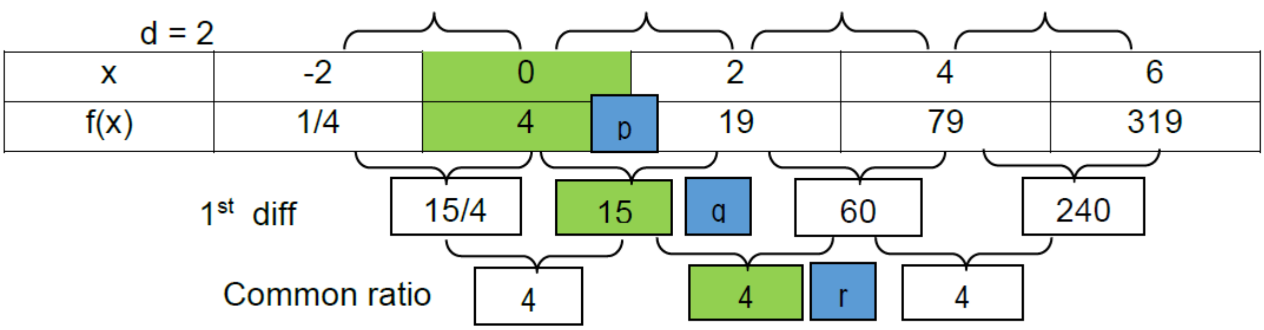

FIGURE 8 Example of a Function of Exponential Trend, d=2

Solution: From the Figure 8 , $d = 2$. At $x = 0$, $p = 4$, $q = 15$, and $r = 4$. Using the formulas, $b = \sqrt[d]{r} = \sqrt[2]{4} = 2,\ a =$

$\frac{q}{b^n(b^d-1)} = \frac{15}{2^0(2^2-1)} = 5$, and $c = p - ab^x = 4 - 5(2)^0 = -1$. Hence, the function described by the given table of ordered pairs is $f(x) = 5(2)^x - -1$.

- **CASE 2:** $d \neq 1$.

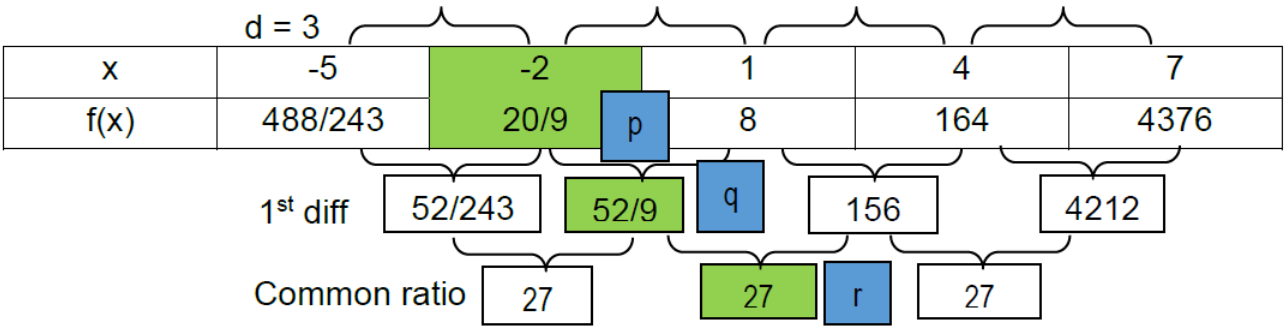

FIGURE 9 Example of a Function of Exponential Trend, d=3

Solution: From the Figure 9 , d = 3. At x = -2, p = 20/9, q = 52/9, and r = 27. Using the formulas, $b = \sqrt[d]{r} = \sqrt[3]{27} = 3$, $a = \frac{q}{b^n(b^d-1)} = \frac{52/9}{3^{-2}(3^3-1)} = 2$,and $c = p - ab^x = \frac{20}{9} - 2(3)^{-2} = 2$. Hence, the function described by the given table of ordered pairs is $f(x) = 2(3)^x + 2$.

4 | CONCLUSION

The study revealed that, (1a) given the equally-spaced values of x in the table of ordered pairs of function, if the first differences of the resulting values of y are equal, then the function is linear. (1b) Given the equally-spaced values of x in the table of ordered pairs of function, if the second differences of the resulting values of y are equal, then the function is quadratic. (1c) An exponential function is a function, where, given a table of ordered pairs, equal differences d in the independent variable x produce a common ratio in the first differences in the dependent variable y. (2) Given the exponential function is f(x) = ab^x + c, then the arbitrary constants have the following formulas: $a = \frac{q}{b^n(b^d-1)}$, $b = \sqrt[d]{r}$, and $c = p - ab^n$, where:

- d is the common difference in the independent variable
- n is any value from the independent variable
- p is the value of the function at $x = n$.
- q is the difference between the values of function at $x = n$ and $x = n + d$.
- r is the common ratio of the values of q.

With the derivation and apparent proofs presented, if the pattern of a function shown in the table of ordered pairs resembles the pattern of $f(x) = ab^x + c$, the derived formulas for the arbitrary constants a, b, and c can be used. Exponential functions have different forms. It can be any of the following: $f(x) = ae^{bx} + c$, $f(x) = e^{ax^2+bx+c}$, $f(x) = ab^x$. It is therefore recommended that future studies should engage in the derivation of the arbitrary constants of the above exponential forms. Since most models developed used the concept of regression, it is recommended that pattern analysis be used specifically when the data can be expressed in terms of ordered pairs.

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